International Journal of Theoretical Physics, Vol. 14, No. 6 (1975), pp. 417–422

Spin-1 Particle in a Homogeneous Magnetic Field

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Received: 9 *June* 1975

Abstract

The Corben–Schwinger theory gives imaginary values of the energy, for $S_3^2 = 1$ states, in very intensive magnetic fields. The theory proposed by the author, which is most satisfactory in the nonrelativistic approximation, does not have this defect for $S_3^2 = 1$ states, but it appears for $S_3^2 = 0$ states.

1. *Introduction*

H. C. Corben & J. Schwinger (1940) have proposed describing the behavior of the spin-1 particle, with anomalous magnetic moment, by a four-vector ψ_r and an antisymmetrical tensor $\psi_{[rs]}$ which satisfy the equations

$$k\psi_r - D^s\psi_{[rs]} + (i\epsilon\lambda/k)B_{[rs]}\,\psi^s = 0 \tag{1.1}$$

$$k\psi_{[rs]} + [D_r\psi_s - D_s\psi_r] = 0 \tag{1.2}$$

with $x_4 = ict$ and

$$k = 2\pi mc/h, \qquad D_r = \partial_r - i\epsilon A_r, \qquad \epsilon = 2\pi q/h \tag{1.3}$$

A_r is the four-potential of the exterior field which acts on the particle and $B_{[rs]} = \partial_r A_s - \partial_s A_r$ is the electromagnetic field. In the nonrelativistic approximation we have shown (Durand, 1976) that this equation involves an electric quadrupole moment and a term with (div E) whose coefficient was not correct. The tensorial equations (1.1), (1.2) are not the most natural when they are written in matrix form. Instead of (1.1), (1.2), we have proposed the equations

$$k\psi_r - D^s\psi_{[rs]} + (i\epsilon\lambda/2k)B_{[rs]}\,\psi^s = 0 \tag{1.4}$$

$$k\psi_{[rs]} + [D_r\psi_s - D_s\psi_r] + (i\epsilon\lambda/2k)[\psi_{[rp]}B_{[s\cdot}{}^{p}{}_{]} - \psi_{[sp]}\,B_{[r\cdot}{}^{p}{}_{]}] = 0 \tag{1.5}$$

In the nonrelativistic approximation equations (1.4) and (1.5) do not have the previously indicated drawback. W. Y. Tsai (1973) has been able to obtain eigenvalues of the energy in the theory of Corben–Schwinger for an external homogeneous magnetic field; he has shown that they become imaginary for the states $S_3^2 = 1$ and for very intensive magnetic fields.

We solve here the same problem using equations (1.4) and (1.5). We shall see that this difficulty is no longer present for states $S_3^2 = 1$, but it appears for the states $S_3^2 = 0$.

Before calculation, we shall replace the set of equations (1.4) and (1.5) by a single partial differential equation of the second order whose resolution is easier.

2. *Partial Differential Equation of the Second Order*

Multiplying (1.4) by D^r and supposing that we are in regions devoid of external charges, where $\partial^r B_{[rs]} = 0$, we obtain

$$D^r \psi_r = -(i\epsilon/2k)B^{[rs]}\psi_{[rs]} - (i\epsilon\lambda/2k^2)B^{[rs]}D_r\psi_s \tag{2.1}$$

Multiplying (1.5) by $B^{[rs]}$ and, on account of

$$B^{[rs]}\,[\psi_{[rp]}B_{[s\cdot}{}^{p}{}_{]} - \psi_{[sp]}B_{[r\cdot}{}^{p}{}_{]}] \equiv 0 \tag{2.2}$$

we get

$$B^{[rs]}\psi_{[rs]} = -(2/k)B^{[rs]}D_r\psi_s \tag{2.3}$$

By substituting (2.3) into (2.1), we obtain

$$D^r\psi_r = (1 - \lambda/2)(i\epsilon/k^2)B^{[rs]}D_r\psi_s \tag{2.4}$$

Still supposing that we are in a region devoid of external charges, we operate on the equation (1.5) from the left by D^s and we obtain

$$kD^s\psi_{[rs]} + i\epsilon B^{[rs]}\psi_s + D_r(D^s\psi_s) - D^sD_s\psi_r + (i\epsilon\lambda/2k)\{B^{[sp]}D_s\psi_{[rp]} - (\partial_s B_{[rp]})\psi^{[sp]} + B_{[rp]}D_s\psi^{[ps]}\} = 0 \tag{2.5}$$

In (2.5) we replace $D^s\psi_{[rs]}$ by its expression (1.4) and $(D^r\psi_r)$ by its expression (2.4). This gives

$$(D^sD^s - k^2)\psi_r = i\epsilon(1+\lambda)B^{[rs]}\psi_s + (1-\lambda/2)(i\epsilon/k^2)D_rB^{[pq]}D_p\psi_q + \frac{\epsilon^2\lambda^2}{4k^2}B_{[pr]}B^{[ps]}\psi_s + \frac{i\epsilon\lambda}{2k}[B^{[sp]}D_s\psi_{[rp]} - (\partial_s B_{[rp]})\psi^{[sp]}] \tag{2.6}$$

One looks now for the particular case of an homogeneous magnetic induction B_w. One has then

$$B_{[w4]} = 0, \qquad B_{[uv]} = B_w = Bn_w, \qquad n_u n^u = 1 \tag{2.7}$$

The three space components of equation (2.6) can be written as

$$(D^s D^s - k^2)\psi_w = i\epsilon(1+\lambda)B_{[wu]}\psi^u + (1-\lambda/2)(i\epsilon/k^2)B^{[uv]}D_w D_u \psi_v$$
$$+ \frac{\epsilon^2\lambda^2}{4k^2}B_{[uw]}B^{[uv]}\psi_v - (i\epsilon\lambda/2k)B^{[uv]}D_u\psi_{[wu]} \qquad (2.8)$$

By introducing the matrices S_u, $\mathscr{E}_{uv}$, ψ, Θ' defined in the paper previously quoted (Durand, 1975), equation (2.8) may be written in the matrix form

$$(D^r D^r - k^2)\psi = \Big\{-\epsilon B(1+\lambda)(S_u n^u) + (\epsilon^2\lambda^2/4k^2)(S_u n^u)^2$$
$$-(1-\lambda/2)\frac{\epsilon B}{k^2}\left[\epsilon B(S_u n^u)^2 + (\mathscr{E}_{uv}D^u D^v)(S_w n^w)\right]\Big\}\psi$$
$$-\frac{i\epsilon\lambda B}{2k}S_u S_v\left[n^u n^v - n^v D^u\right]\Theta' \qquad (2.9)$$

The expression for Θ', given in Durand (1976), reduces here to

$$\Theta' = (i/k)(1+K)(S_u D^u)\psi \qquad (2.10)$$

with

$$K = a(S_u n^u) + b(S_u n^u)^2 \qquad (2.11)$$

$$a = \frac{\lambda\epsilon'\xi}{(1-\lambda^2\xi^2)}, \qquad b = \frac{\lambda^2\xi^2}{(1-\lambda^2\xi^2)}, \qquad \xi = \frac{\omega h}{4\pi mc^2} \qquad (2.12)$$

$$\omega = |q|B/m, \qquad \epsilon' = q/|q| \qquad (2.13)$$

Bringing (2.10) into (2.11), we obtain

$$(D^r D_r - k^2)\psi = \{-\epsilon B(1+\lambda)(\mathbf{S}\cdot\mathbf{n}) + (\epsilon^2\lambda^2B^2/4k^2)(S_u n^u)^2$$
$$-(\epsilon B/k^2)(1-\lambda/2)[\epsilon B(\mathbf{S}\cdot\mathbf{n}) + (\mathbf{D})^2 - (\mathbf{S}\cdot\mathbf{D})^2[(\mathbf{S}\cdot\mathbf{n})$$
$$+(i\epsilon B\lambda/2k^2)(\mathbf{S}\cdot[\mathbf{n}\times\mathbf{D}])(1+K)(\mathbf{S}\cdot\mathbf{D})\}\psi \qquad (2.14)$$

which is the equation we were seeking.

3. *Eigenvalues of the Energy*

Let us consider the particular case $\mathbf{n} = (0, 0, 1)$ and $D_3 = 0$. We have then a magnetic field in the z direction and ψ is independent of z. If, moreover, we assume an exponential time dependence, we have also

$$(\mathbf{n}\cdot\mathbf{S}) = S_3, \qquad D_4 = (1/ic)\partial_t = -(2\pi/ch)W \qquad (3.1)$$

$$D_4^2 = (2\pi/h)^2(W/c)^2 = k^2(W/mc^2)^2 \qquad (3.2)$$

Let us introduce the operators Q and R, defined by

$$Q = (S_1^2 - S_2^2)(D_1^2 - D_2^2) + (S_1S_2 + S_2S_1)(D_1D_2 + D_2D_1) \quad (3.3)$$

$$R = D_1^2 + D_2^2 + 2\epsilon BS \quad (3.4)$$

These operators, given by W. Y. Tsai (1973) have noteworthy properties. In the first place, Q anticommutes with S_3, that is

$$\{Q, S_3\} = 0 \quad (3.5)$$

Consequently, it commutes with S_3^2

$$[Q, S_3^2] = 0 \quad (3.6)$$

Moreover, we have

$$S_3^2 Q = QS_3^2 = Q \quad (3.7)$$

Finally, Q commutes with R

$$[Q, R] = 0 \quad (3.8)$$

and the square of Q involves S_3^2 and R^2; more precisely

$$Q^2 = S_3^2(R^2 - \epsilon^2 B^2) \quad (3.9)$$

Using the properties of the matrices S_u, one finds the expressions

$$i(\mathbf{S} \cdot [\mathbf{n} \times \mathbf{D}])(\mathbf{S} \cdot \mathbf{D}) = \tfrac{1}{2}\{[R - 3\epsilon BS_3 - Q]S_3 + 2\epsilon B\} \quad (3.10)$$

$$i(\mathbf{S} \cdot [\mathbf{n} \times \mathbf{D}])S_3(\mathbf{S} \cdot \mathbf{D}) = -(1 - S_3^2)R = -(1 - S_3^2)(D_1^2 + D_2^2) \quad (3.11)$$

$$i(\mathbf{S} \cdot [\mathbf{n} \times \mathbf{D}])S_3^2(\mathbf{S} \cdot \mathbf{D}) = \epsilon B(1 - S_3^2) \quad (3.12)$$

$$[\epsilon B \cdot (\mathbf{S} \cdot \mathbf{n}) + (D)^2 - (\mathbf{S} \cdot \mathbf{D})^2](\mathbf{S} \cdot \mathbf{n}) = \tfrac{1}{2}\{R - \epsilon BS_3 - Q\}S_3 \quad (3.13)$$

By introducing (3.10)-(3.13) into (2.14) and owing to (2.11)-(2.13), we obtain

$$\begin{aligned} k^2(W/mc^2)^2\psi = \{k^2 &- (D_1^2 + D_2^2) - 2k^2\epsilon'(1+\lambda)\xi S_3 \\ &+ k^2\xi^2\lambda^2 S_3^2 - \epsilon'\xi(1-\lambda)[R - 2k^2\epsilon'\xi S_3 - Q]S_3 \\ &+ [2k^2\xi^2\lambda - \lambda^2\xi^2(D_1^2 + D_2^2)(1 - S_3^2)/(1 - \lambda^2\xi^2)\}\psi \end{aligned} \quad (3.14)$$

The operator S_3^2 commutes with all the operators that one finds in (3.14). One can then choose, for ψ, eigenfunctions of S_3^2 whose eigenvalues are zero or one.

If $\boxed{S_3^2 = 0}$ one has also $S_3 = 0$ and equation (3.14) reduces to

$$(W/mc^2)^2\psi = \psi - \{(1/k^2)(D_1^2 + D_2^2) - 2\xi^2\lambda\}\psi/(1 - \lambda^2\xi^2) \quad (3.15)$$

But the eigenvalues of $-(D_1^2 + D_2^2)$ are

$$(2N + 1)2\xi k^2 \quad (3.16)$$

with $N = 0, 1, 2, 3, \ldots$. Substituting (3.16) into (3.15), one gets

$$(W/mc^2)^2 = 1 + [(2N+1)2\xi + 2\lambda\xi^2]/(1 - \lambda^2\xi^2) \qquad (3.17)$$

from which one can obtain the energy W.

If $\boxed{S_3^2 = 1}$, equation (3.14) gives

$$\left(\frac{W}{mc^2}\right)^2 \psi = \left\{1 - \frac{R}{k^2} + 2(1-\lambda)\xi^2 + \xi^2\lambda^2 + \frac{\epsilon'\xi}{k^2}(1-\lambda)S_3[2k^2 - R - Q]\right\}\psi \qquad (3.18)$$

The operator Q does not commute with S_3 and it must be eliminated. For this, one may use the canonical transformation given by W. Y. Tsai dealing with the Corben-Schwinger theory.

It proceeds from the following considerations: We consider three operators A, B, C which commute except C, that does not commute with B but anti-commutes ($\{C, B\} = 0$) and we consider their combination

$$C(A + B)$$

We consider also the operator T and its inverse T^{-1}, such that

$$T^{\pm 1} = (1/\sqrt{2})\{\lambda_{(+)} \pm (B/|B|)\lambda_{(-)}\} \qquad (3.19)$$

with

$$\lambda_{(+)} = \sqrt{\frac{A}{\sqrt{A^2 - B^2}} + 1}, \qquad \lambda_{(-)} = \sqrt{\frac{A}{\sqrt{A^2 - B^2}} - 1} \qquad (3.20)$$

We can verify, with $\lambda_{(+)}\lambda_{(-)} = |B|/\sqrt{A^2 - B^2}$ that we have

$$2TC(A + B)T^{-1} = C\sqrt{A^2 - B^2} \qquad (3.21)$$

Under these conditions, we perform the canonical transformation $\psi' = T\psi$, in (37), with

$$C \to S_3, \qquad A \to (2k^2 - R), \qquad B \to -Q \qquad (3.22)$$

On account of (3.9) and

$$\sqrt{(2k^2 - R)^2 - Q^2} = 2k^2\sqrt{1 + \xi^2 - R/k^2} \qquad (3.23)$$

equation (3.18) may be written

$$\begin{aligned}(W/mc^2)^2\psi' &= \{1 - R/k^2 \\ &\quad + 2(1-\lambda)\xi^2 + \xi^2\lambda^2 + 2\epsilon'\xi(1-\lambda)S_3\sqrt{1 + \xi^2 - R/k^2}\}\psi' \\ &= \{\sqrt{1 + \xi^2 - R/k^2} + \epsilon' S_3\xi(1-\lambda)\}^2\psi' \qquad (3.24)\end{aligned}$$

because T commutes with operators other than operators A, B, C. This equation (3.24) contains only operators S_3 and R which commute. One can then choose for ψ' an eigenfunction common to these two operators. On account of (3.16) and with the eigenvalues of S_3 equal to $\mu = \pm 1$, one has

$$W/mc^2 = \sqrt{1 + \xi^2 + (2N + 1 - 2\epsilon'\mu)2\xi} + \epsilon'\mu\xi(1 - \lambda) \tag{3.25}$$

For small values of ξ, with $\lambda = 1 + 2\kappa$, this equation (3.25) reduces to

$$W/mc^2 \neq 1 + [2N + 1 - 2\epsilon'\mu(1 + \kappa)]\, 2\xi \tag{3.26}$$

One then recovers the result of the nonrelativistic theory of the spin, for a particle whose gyromagnetic ration is $g = 2(1 + \kappa)$. The Corben-Schwinger theory, instead of (3.17) and (3.25), leads respectively to

$$W/mc^2 = 1 + (2N + 1)2\xi \tag{3.27}$$

and

$$\begin{aligned}(W/mc^2)^2 &= 1 + (2N + 1 - 2\epsilon'\mu)2\xi + 2(1 - \lambda)\xi^2 \\ &\quad + 2\epsilon'\xi(1 - \lambda)S_3\sqrt{1 + \xi^2 + (2N + 1 - 2\epsilon'\mu)2\xi} \qquad (3.28\end{aligned}$$

In equation (3.28), the term $\xi^2\lambda^2$ which appeared in equation (3.24) does not appear, the latter being a perfect square. Consequently the right-hand side of (3.28) may become negative and the energy W may be imaginary. On account of this, W. Y. Tsai says that the theory of Corben-Schwinger is "inconsistent." On the other hand formula (3.27) which corresponds to states $S_3^2 = 0$, does not have this defect. Inversely, our theory leads to a satisfactory expression for the states $S_3^2 = 1$, but the formula (3.17) which concerns the states $S_3^2 = 0$ is not convenient. Not only may the energy become imaginary, but it becomes infinite for $\xi^2\lambda^2 = 1$. One can also say that the states $S_3^2 = 0$ no longer exist when $\xi^2\lambda^2 > 1$.

References

Corben, H. C. and Schwinger, J. (1940). *Physical Review,* **58**, 953.
Durand, E. (1975). *Il nuovo Cimento* **28B**, 415.
Tsai, W. Y. (1973). *Physical Review D,* **7**, 1945.